Advanced Quantum Field Theory (2024/2025)
TP 4 - Quantum effective action for ϕ^3 theory in $d = 6$

The quantum effective action $\Gamma[\phi]$ is defined as the Legendre transform

$$\Gamma[\phi] = (W[J] - \int d^d x J\phi)|_{J_\phi} \qquad \frac{\delta W[J]}{\delta J} = \phi \Leftrightarrow J = J_\phi \tag{1}$$

where $W[J]$ is the generating functional of connected diagrams

$$Z[J] = e^{iW[J]} \,. \tag{2}$$

For the action of a scalar field of the form

$$S[\phi] = \int d^4x \Big(-\frac{1}{2}\partial_\mu \phi_A \partial^\mu \phi_A - \frac{1}{2} m^2 \phi_A^2 - V[\phi_A] \Big) \,, \tag{3}$$

the *1-loop quantum effective action* takes the form

$$\Gamma[\phi] = S[\phi] + \hbar \Gamma^{(1)}[\phi] = S[\phi] - \frac{\hbar}{2i} \mathrm{Tr} \ln \Big(\delta^A_B \delta^{(4)}(x,y) + \mathcal{D}^{-1AB}(x,y) V''_{AB}[\phi_A] \Big) + \mathcal{O}(\hbar^2) \,, \tag{4}$$

where $\mathcal{D}^{-1AB}(x,y)$ is the Green function for the Klein–Gordon operator $\delta_{AB}(-\partial^2 + m^2)$ and V''_{AB} is the second variation of the potential. Using the expansion of the logarithm, the one-loop contribution $\hbar\Gamma^{(1)}[\phi]$ can be written as

$$\Gamma^{(1)}[\phi] = -\frac{1}{2i} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \int d^d x_1 \dots d^d x_n V''[\phi(x_1)] \dots V''[\phi(x_n)]$$
$$\int \frac{d^d q_2}{(2\pi)^d} \cdots \frac{d^d q_n}{(2\pi)^d} e^{-ix_1(q_2 + \ldots + q_n)} e^{ix_2 q_2} \dots e^{ix_n q_n} \gamma^{(n)}(q_2, \dots q_n) \,, \tag{5}$$

where

$$\gamma^{(n)}(q_2, \dots q_n) = \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + m^2 - i\epsilon} \frac{1}{(q+q_2)^2 + m^2 - i\epsilon} \cdots \frac{1}{(q + q_2 + \ldots + q_n)^2 + m^2 - i\epsilon}, \tag{6}$$

and we restricted for simplicity to the case of a single scalar field.

Exercise 1: One-loop renormalization of ϕ^3 theory in six dimensions

We want to use the one-loop effective action to renormalize ϕ^3-theory in $d = 6$ dimensions.

1. Based on power counting, which integrals $\gamma^{(n)}$ are divergent in $d = 4$ dimensions and $d = 6$ dimensions?

2. Using dimensional regularization, determine the divergent part of these integrals in $d = 6 - 2\epsilon$ dimensions. You should find

$$\gamma^{(1)}_\infty = \frac{im^4}{2(4\pi)^3} \frac{1}{\epsilon}, \qquad \gamma^{(2)}_\infty = -\frac{i}{(4\pi)^3} \frac{1}{\epsilon} \Big(\frac{1}{6} q_2^2 + m^2 \Big), \qquad \gamma^{(3)}_\infty = \frac{i}{2(4\pi)^3} \frac{1}{\epsilon} \,, \tag{7}$$

   Hint: Useful formulas can be found on the last page. Before evaluating $\gamma^{(3)}$ using Feynman parameters, think about the form of the final answer.

3. Consider now the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0 - \frac{1}{2}m_0^2\phi_0^2 - \frac{g_0}{3!}\phi_0^3 + \kappa_0\phi_0, \tag{8}$$

in d dimensions. Compute the dimensions of ϕ_0, m_0, g_0, and κ_0. Argue that this theory is renormalizable in $d = 6$ dimensions. We want to set-up renormalized perturbation theory. Define renormalized quantities by

$$\phi_0 = Z_\phi^{1/2}\phi_R \qquad m_0^2 = Z_m m_R^2 \qquad g_0 = Z_g g_R\mu^\alpha \qquad \kappa_0 = \kappa_R Z_\kappa\,, \tag{9}$$

with α chosen such that g_R is dimensionless. Assume that the renormalization factors can be expanded as $Z_i = 1 + \delta_i$ where δ_i is $O(g_R^2)$.

4. Show that the divergent contributions of the 1-loop effective action can be eliminated by appropriately choosing δ_i.

$$S[\phi] + \Gamma_\infty^{(1)}[\phi] = \int d^4x\left(-\frac{1}{2}\partial_\mu\phi_R\partial^\mu\phi_R - \frac{1}{2}m_R^2\phi_R^2 - \frac{g_R}{3!}\phi_R^3 + \kappa_R\phi_R,\right). \tag{10}$$

You should find

$$\phi_R = Z^{-1/2}\phi_0, \qquad Z = 1 - \frac{g_R^2}{12(4\pi)^3\epsilon} \tag{11a}$$

$$m_0^2 = \left(1 - \frac{5g_R^2}{12(4\pi)^3\epsilon}\right)m_R^2 \tag{11b}$$

$$g_0 = \left(1 - \frac{3g_R^2}{8(4\pi)^3\epsilon}\right)g_R \tag{11c}$$

$$\kappa_0 = \left(1 + \frac{g_R^2}{24(4\pi)^3\epsilon}\right)\kappa_R + \frac{g_R m_R^4\mu^{-\epsilon}}{4(4\pi)^3\epsilon}\,. \tag{11d}$$

We have therefore shown that the theory is *renormalizable* at one-loop level.

5. Compute the beta function from the above relation between bare and renormalized coupling. You should find

$$\beta(g_R) = -\frac{3g_R^3}{256\pi^3}\,. \tag{12}$$

What does this relation tell us about ϕ^3-theory in $d = 6$?

6. In the case of ϕ^3 theory in $d = 6$ we were able to absorb all divergent contributions of the 1-loop effective action in a redefinition of the parameters of the bare action. Consider now a theory with interaction term ϕ^4 in $d = 6$ and determine the functional form of the divergent terms generated by the 1-loop effective action. Are we still able to get rid of all divergencies by renormalization of the bare parameters? Compute the dimension of the coupling constant and conclude that this theory is *non-renormalizable*, i.e., quantum corrections force us to add an infinite number of additional terms in order to cancel all divergences.

Useful formulas

Feynman parametrization

$$\frac{1}{A_1\ldots A_n} = (n-1)!\int_0^1 dx_1\ldots\int_0^1 dx_n\,\delta(x_1+\ldots+x_n-1)\frac{1}{(A_1x_1+A_2x_2+\ldots+A_nx_n)^n}\,. \tag{13}$$

The integral

$$\int\frac{d^dq}{(2\pi)^d}\frac{1}{(q^2+\Delta^2-i\epsilon)^a} = i\int\frac{d^dq_E}{(2\pi)^d}\frac{1}{(q_E^2+\Delta^2)^a} = i\Phi(\Delta,d,a) \tag{14}$$

yields

$$\Phi(\Delta, d, a) = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(a - d/2)}{\Gamma(a)} \left(\frac{1}{\Delta^2} \right)^{a-d/2} . \tag{15}$$

The $\Gamma(x)$ function has a simple pole at $x = 0$

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E \, , \tag{16}$$

where γ_E is the Euler–Mascheroni constant.